The effective displacement thickness, defined by Eqs. (15 and 16), differs from the one defined by Eqs. (5b and 2) only by terms of order $(\delta - \Delta^*)^2 \sim 1/R$. Terms of this order are neglected in the preceding analysis.

In three-dimensional flows it becomes necessary to work with the individual velocity components rather than the stream function. For boundary-layer flow about a general body, the surface of which is represented by $y(x, z) = 0$, the normal velocity components in the outer (inviscid) and inner (boundary-layer) expansions are given by

$$v(x, z, y;\ R) \sim V_1(x, z, y) + \frac{1}{R^{1/2}} V_2(x, z, y) + \ldots \quad (17)$$

$$v(x, z, y;\ R) \sim \frac{1}{R^{1/2}} v_1(x, z, \bar{y}) + \frac{1}{R} v(x, z, \bar{y}) + \ldots \quad (18)$$

The leading term in the outer expansion vanishes at $y = 0$. The second term represents the perturbation due to the boundary-layer displacement effect. As demonstrated by Davis and Flügge-Lotz,[5] there are two different ways of matching the inner and outer variables. The results are listed below

$$V_{1_y}(x, z, 0) = \lim_{\bar{y}\to\infty} v_{1_{\bar{y}}}(x, z, \bar{y}) \quad (19)$$

$$V_2(x, z, 0) = \lim_{\bar{y}\to\infty} [v_1(x, z, \bar{y}) - \bar{y} v_{1_{\bar{y}}}(x, z, \bar{y})] \quad (20)$$

The effective displacement thickness now may be introduced as the thickness Δ^* of a layer that, when added to the body, produces an inviscid normal velocity equal to $\frac{1}{R^{1/2}} V_2$ near the surface.

$$\mathbf{q}\cdot\text{grad } \Delta^* = \Delta^* \left.\frac{\partial V_1}{\partial y}\right|_{y=0} + \frac{1}{R^{1/2}} V_2(x, z, 0) + 0\left(\frac{1}{R}\right) \quad (21)$$

By means of the perturbation expansion for the inviscid velocity vector $\mathbf{q}$ (i.e., $\mathbf{q} = \mathbf{q}_0 + \Delta^* \cdot \mathbf{q}_{0_y} \ldots + R^{1/2}\mathbf{q}_1 + \ldots$), and Eq. (19), the forementioned equation can be rewritten as follows: $(\bar{\Delta}^* = R^{1/2}\Delta^*)$

$$\mathbf{q}_0\cdot\text{grad } \Delta^* = \frac{1}{R^{1/2}} \times \left\{ \lim_{\bar{y}\to\infty} [v_1(x, z, \bar{y}) - (\bar{y} - {}^*)\bar{\Delta} v_{1_{\bar{y}}}(x, z, \bar{y})] \right\} \quad (22)$$

This equation, which is equivalent to the relation derived by Moore,[7] defines the effective displacement thickness Δ^*. It is equally valid both with and without mass transfer at the surface, but in general it does not lead to explicit relations for Δ^*. A much more useful result can be obtained by a simple extension of a method developed by Lighthill.[8] He makes use of streamline coordinates, but this is hardly a drawback since these are used almost exclusively nowadays in three-dimensional boundary-layer calculations.

Lighthill considers the reduction in mass flow in the boundary-layer region bounded by the normal planes through two neighboring streamlines located a distance $h_z dz$ apart. The mass flow defect in the x direction is

$$(h_z dz) \int_0^\infty (\rho_e u_e - \rho u) d\bar{y}$$

The flow in the boundary layer is not parallel to the external streamlines and the total loss of mass through the sides from the point of attachment $x = 0$ equals

$$\frac{\partial}{\partial z}\left[\int_0^x h_x\, dx \int_0^\infty \rho w d\bar{y}\right] dz$$

The total mass flow added to the region considered is

$$\int_0^x (\rho_w v_w h_z h_x dz) dx$$

The accumulation of mass in the boundary layer can be accounted for by an equivalent displacement Δ^* of the inviscid streamlines such that

$$(h_z dz)\rho_e u_e \Delta^* = (h_z dz) \int_0^\infty (\rho_e u_e - \rho u) dy + \int_0^x (\rho_w v_w h_z h_x dx) dz - \frac{\partial}{\partial z}\left[\int_0^x h_x\, dx \int_0^\infty \rho w d\bar{y}\right] dz$$

By rearranging one obtains

$$\Delta^* = \delta^* + \frac{1}{\rho_e u_e h_z} \int_0^x \rho_w v_w h_z h_x dx - \frac{1}{\rho_z u_e h_z} \cdot \frac{\partial}{\partial z}\left[\int_0^x h_x\, dx \int_0^\infty \rho w\, d\bar{y}\right] \quad (23)$$

In many cases of practical interest the cross flow is very small and the last term may be neglected. One notes that Eq. (23) reduces to Eq. (15) in the case of axisymmetric flows ($h_z = r_w$) whenever $\delta \ll r_w$.

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Reply to T. K. Fannelop

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Li and Gross obtained an expression of v_e/u_e in Eq. (10) of Ref. 1. This expression is correct.

For 2-dimensional flow, Eq. (10) of Ref. 1 becomes

$$\frac{v_e}{u_e} = \frac{\rho_w v_w}{\rho_e u_e} + \frac{d\delta^*}{dx} - \frac{\delta - \delta^*}{\rho_e u_e} \frac{d}{dx} (\rho_e u_e)$$

Equation (A) of Ref. 2 states, on the other hand,

$$\frac{v_e}{u_e} = \frac{\rho_w v_w}{\rho_e u_e} + \frac{d\delta^*}{dx} - \frac{\Delta^* - \delta^*}{\rho_e u_e} \frac{d}{dx} (\rho_e u_e)$$

This discrepancy constitutes an interesting question of the proper choice of the boundary conditions in viscous interaction theory. I invited Professor O. R. Burggraf's atten-

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tion to this matter and have had the benefit of his discussions. In Ref. 3, it has been shown that the preceding results of Li and Gross and of Fannelop can be treated as special applications within a general formulation.

Li and Gross in Ref. 1 did not propose a formula such as Eq. (4) of Ref. 2. Some of Fannelop's statements (after Eq. (4) and before Eq. (5b) of Ref. 2) properly should leave out the names of Li and Gross.

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Choice of Boundary Conditions in Viscous Interaction Theory

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In the viscous interaction problem, the outer inviscid flow is affected by the growth of the boundary layer. The magnitude of this interaction is computed either by recomputing the inviscid flow following a boundary-layer calculation, as for weak interactions, or by computing both viscous and inviscid flows simultaneously, as for strong interactions. In either case the boundary condition to be applied to the inviscid flow must account properly for the growth of the boundary layer. In a previous note to this Journal, Li and Gross (Ref. 1), as well as other authors, have given this boundary condition as a formula for the effective blowing velocity at the outer edge of the viscous layer.† In opposition, Fannelop (Ref. 2) introduces a displacement thickness defined as the distance from the surface to the zero streamline of the perturbed inviscid flow and shows that the effective blowing velocity obtained from the slope of this displacement thickness differs from that given in Ref. 1. By implication, the formula of Li and Gross should be incorrect, and consequently, any analyses based on this formula would be invalid. Actually the results of Fannelop and of Li and Gross are equivalent when interpreted properly, as is shown by the following analysis. For simplicity, we restrict ourselves to two-dimensional flow.

As in Fannelop's analysis, the outer (inviscid flow) and inner (viscous flow) expansions of the stream function are expressed in the physical coordinates (x, y) and the boundary-layer variables $(x, \bar{y})$, where $\bar{y} = y(R)^{1/2}$ in the form

$$\psi(x, y) = \psi_1(x, y) + [1/(R)^{1/2}]\psi_2(x, y) + (1/R)\psi_3(x, y) + \dots \quad (1)$$

and

$$\psi(x, y) = [1/(R)^{1/2}]\bar{\psi}_1(x, \bar{y}) + (1/R)\bar{\psi}_2(x, \bar{y}) + \dots \quad (2)$$

The body is presumed smooth everywhere so that no logarithmic terms appear in the expansions. The mass flux from the surface $\rho_w v_w$ is of order $1/(R)^{1/2}$ to permit an attached boundary layer. Thus ψ_1 satisfies the boundary condition $\psi_1(x, 0) = 0$. Both ψ_1 and ψ_2 satisfy inviscid flow equations, but diffusion of external vorticity begins with the third order term ψ_3.

Since the matching conditions are crucial to the argument, we repeat the matching procedure here to permit our reinterpretation. The basic idea is that the outer expansion in the vicinity of the wall should match smoothly, to the order of the terms involved, with the inner expansion in the outer fringes of the boundary layer (or inner region). Consequently, a Taylor series representation of the outer expansion, when rewritten in terms of boundary-layer variables, should match term-by-term with an asymptotic expansion of the inner expansion at large $\bar{y}$. Thus

$$\begin{aligned}\psi(x, y) &= \psi(x, 0) + y\psi_y(x, 0) + \tfrac{1}{2}y^2\,\psi_{yy}(x, 0) + \dots \\ &= 0 + [1/(R)^{1/2}]\bar{y}\,\psi_y(x, 0) + (1/2R)\bar{y}^2\,\psi_{yy}(x, 0) + \dots \\ &= [1/(R)^{1/2}][\bar{y}\,\psi_{1y}(x, 0) + \psi_2(x, 0)] + \\ &\quad (1/R)[\tfrac{1}{2}\bar{y}^2\,\psi_{1yy}(x, 0) + \bar{y}\,\psi_{2y}(x, 0) + \psi_3(x, 0)] + \dots \end{aligned} \quad (3)$$

Comparing like terms in $1/R$ in (3) with the inner expansion (2), we obtain the matching conditions

$$\lim_{\bar{y}\to\infty} [\bar{\psi}_1(x, \bar{y}) - \bar{y}\,\psi_{1y}(x, 0) - \psi_2(x, 0)] = 0 \quad (4)$$

$$\lim_{\bar{y}\to\infty} [\bar{\psi}_2(x, \bar{y}) - \tfrac{1}{2}\bar{y}^2\,\psi_{1yy}(x, 0) - \bar{y}\,\psi_{2y}(x, 0) - \psi_3(x, 0)] = 0 \quad (5)$$

and in a similar way, higher order matching conditions are obtained. (Note that the term $\psi_{1yy}(x, 0)$ in (5) is associated with external vorticity.) Our present interests are restricted to terms of order $1/(R)^{1/2}$ corresponding to matching condition (4). For large $\bar{y}$, we may express the asymptotic form of $\bar{\psi}$, as

$$\bar{\psi}_1(x, \bar{y}) \sim \psi_{1y}(x, 0)[\bar{y} - \alpha(x)] \quad (6)$$

where the terms neglected are exponentially small in $\bar{y}$. The expansion (6) satisfies condition (4) provided

$$\bar{\psi}_{1\bar{y}}(x, \bar{y}) \underset{\bar{y}\to\infty}{\longrightarrow} \psi_{1y}(x, 0) \quad (7)$$

and

$$\psi_2(x, 0) = -\alpha(x)\,\psi_{1y}(x, 0) \quad (8)$$

As Fannelop points out, Eq. (7) is just the outer boundary condition in Prandtl's boundary-layer theory. However, the matching condition (4) also yields the basic boundary condition on ψ_2, given by Eq. (8). A numerical value for $\alpha(x)$ may be obtained by an asymptotic analysis of the boundary-layer solution. However, by comparing (8) with Fannelop's equation (10), and noting $\psi_1(x, 0) = 0$, $\alpha(x)$ is identified with Fannelop's Δ^*. Hence, we have

$$\frac{\alpha(x)}{(R)^{1/2}} = \Delta^* = \delta^* + \frac{1}{\rho_e u_e}\int_0^x \rho_w v_w dx \quad (9)$$

where $\rho_e u_e = [1/(R)^{1/2}]\psi_{1y}(x, 0)$, δ^* is the conventional displacement thickness and the terms neglected are of order $1/R$. Then the boundary condition (8) is given alternatively as

$$\psi_2(x, 0) = -(R)^{1/2}\left[\rho_e u_e \delta^* + \int_0^x \rho_w v_w dx\right] \quad (10)$$

The latter form for $\psi_2(x, 0)$ is convenient if the boundary layer is calculated approximately by a momentum-integral method, or more simply by local similarity.

The effective blowing velocity at the surface, accounting for the boundary layer growth, now is obtained from (10):

$$[\rho_e v_e]_{y=0} = -[1/(R)^{1/2}]\psi_{2x}(x, 0) = \rho_w v_w + \rho_e u_e(d\delta^*/dx) + \delta^*(d/dx)(\rho_e u_e) \quad (11)$$

where again neglected terms are of order $1/R$.

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† These authors were concerned with the effects of transverse curvature of a surface with mass transfer.